Published by Institute of Physics Publishing for SISSA/ISAS
Received: *November 20, 2003*
Accepted: *December 12, 2003*

$\mathrm{SL}(2,\mathbb{Z})$ action on three-dimensional CFTs and holography

Robert G. Leigh

Department of Physics, University of Illinois at Urbana-Champaign
Urbana, IL 61801 U.S.A.
Email: rgleigh@uiuc.edu

Anastasios C. Petkou

CERN Theory Division
CH-1211 Geneva 23, Switzerland
Email: tassos.petkou@cern.ch

Abstract: We show that there is a natural action of $\mathrm{SL}(2,\mathbb{Z})$ on the two-point functions of the energy momentum tensor and of higher-spin conserved currents in three-dimensional CFTs. The dynamics behind the S-operation of $\mathrm{SL}(2,\mathbb{Z})$ is that of an irrelevant current-current deformation and we point out its similarity to the dynamics of a wide class of three-dimensional CFTs. The holographic interpretation of our results raises the possibility that many three-dimensional CFTs have duals on AdS_4 with $\mathrm{SL}(2,\mathbb{Z})$ duality properties at the linearized level.

Keywords: Duality in Gauge Field Theories, AdS-CFT and dS-CFT Correspondence.

JHEP12(2003)020

 http://jhep.sissa.it/archive/papers/jhep122003020/jhep122003020.pdf

Contents

1. Introduction

Conformal field theories in spacetime dimensions $d > 2$ are relatively rare entities. In the particularly interesting case of four spacetime dimensions, it seems possible that non-trivial CFTs require supersymmetry, while the majority of them do not have a simple lagrangian formulation. Nevertheless, it has been known for a long time that in three dimensions there exists a number of non-trivial CFTs that do not necessarily require supersymmetry, while at the same time they do have a simple lagrangian formulation that allows the evaluation of dynamical quantities like anomalous dimensions. Such three-dimensional CFTs have a wealth of interesting perturbative as well as non-perturbative properties that appear to be physically relevant as they describe universality classes of real statistical systems (for a recent review containing a large list of references see [1]).

The recent success of ideas related to holography leads naturally to question whether the rich web of three-dimensional CFTs could be related to quantum theories on (A)dS$_4$ spaces. This simple question turns out to be unexpectedly important in view of the recent interest in quantum field theory on four-dimensional spaces with cosmological constant. There is, however, an additional reason to be interested in the holographic properties of three-dimensional CFTs. It is known that in many of the latter theories one has control on both the weak-coupling (free field theory limit) and the strong-coupling regimes. This property provides the right framework to study the holography of free field theories, a subject that has recently attracted attention [2, 3, 4].

JHEP12(2003)020

A bold proposal for a holographic dual of a three-dimensional CFT was first made in [5] for the Critical $O(N)$ Vector model[1] and and its connection with a Higher-Spin theory[2] on AdS_4. Soon thereafter the Higgs mechanism giving masses to the Higher-Spins in AdS_4 was discussed [9]. In [10] we have argued that such a proposal could lead to the reconstruction of a consistent quantum theory on AdS_4. Moreover, it was further realized in [11, 12] that both the Critical $O(N)$ Vector model and the fermionic $O(N)$ Gross-Neveu model can be contained in the holographic dual of a unique $\mathcal{N}=1$ supersymmetric Higher-Spin theory on AdS_4.

An interesting yet apparently unrelated observation was recently made by Witten in [13] (see also [14, 15, 17]). There it was shown that there is a natural $\mathrm{SL}(2,\mathbb{Z})$ action on three-dimensional CFTs with U(1) conserved currents. This action consists of a sequence of operations, described in detail in [13], that lead from a given CFT to its dual. The relevant point for us is that the S- and T-operations of $\mathrm{SL}(2,\mathbb{Z})$ act on the two-point function of the conserved spin-1 current of the initial CFT and produce the two-point function of the spin-1 conserved current of the dual CFT. Quite intriguingly, it was further argued in [13] that the $\mathrm{SL}(2,\mathbb{Z})$ action on the three-dimensional CFT may be viewed as the holographic image of the well-known $\mathrm{SL}(2,\mathbb{Z})$ duality of pure electrodynamics on AdS_4.

In this work we show that the $\mathrm{SL}(2,\mathbb{Z})$ action naturally extends to two-point functions of the energy momentum tensor and of higher-spin conserved currents in three-dimensional CFTs. In particular, we show that the S-operation of $\mathrm{SL}(2,\mathbb{Z})$ on two-point functions is implemented by coupling the corresponding current to an external field, evaluating its induced propagator and reading from the latter the two-point function of an appropriately defined dual conserved current. We present the calculation for the energy momentum tensor (spin-2 current) in some detail and sketch the corresponding calculation for higher-spin conserved currents. The T-operation of $\mathrm{SL}(2,\mathbb{Z})$ acts as in [13] by shifting the coefficient of the conformally invariant contact term which appears in the two-point functions of all of these conserved currents.

Moreover, we show that the S-operation on the two-point functions of conserved currents may be implemented dynamically by certain double-trace deformations.[3] We point out the similarity of these double-trace deformations to the dynamics of some well-known three-dimensional CFTs, such as the $O(N)$ Vector and Gross-Neveu models and the Thirring model.

Then we look for a holographic interpretation of our $\mathrm{SL}(2,\mathbb{Z})$ action. This leads us naturally to consider theories on AdS_4 that are generalizations of the self-dual pure electrodynamics, at least at the linearized level. For gravity, we argue that such a theory may be given by the old MacDowell-Mansouri action [18] with the addition of the topological Pontryagin term that plays the role of a θ-term [19, 20]. We show that this action gives rise to the two coefficients in the two-point function of the boundary energy momentum tensor, in the same way as pure electromagnetism gives rise to the two coefficients in the two-point

[1]Earlier ideas for a connection between free CFTs and Higher-Spins are recorded in [6].

[2]For a recent work on Higher-Spins with an extensive reference list see [7, 8].

[3]In a slight abuse of terminology, we keep here the notion of "double-trace" composite operators in theories with elementary $O(N)$ vector fields in par with recent literature.

function of the boundary spin-1 current. For higher-spins, the relevant AdS_4 action may be the recently discussed action of Vasiliev [21] with the addition of the appropriate Pontryagin term for higher-spin curvatures of $SO(3,2)$. Therefore, we raise the possibility that a number of three-dimensional CFTs may have AdS_4 duals with $SL(2,\mathbb{Z})$ duality properties, at least at the linearized level. This also emphasizes the intimate relation of $SL(2,\mathbb{Z})$ duality to the holography of free field theories.

The paper is organized as follows. In section 2 we demonstrate the S-operation on two-point functions of higher-spin conserved currents. In section 3 we show that the S-operation can be implemented by double-trace deformations, in close analogy to the dynamics of some well-known three-dimensional CFTs. In section 4 we discuss the holographic interpretation of our results and in particular a gravitational action that may induce our $SL(2,\mathbb{Z})$ action in the boundary. We conclude and discuss further implications of our results in section 5.

2. $SL(2,\mathbb{Z})$ action on two-point functions of conserved currents

2.1 Review of the $U(1)$ current case

We begin with a review of Witten's result [13] (see also [22]) for the two-point function of a conserved spin-1 current. With the definition

$$\Pi^{\mu\nu}(p) = p^\mu p^\nu - \delta^{\mu\nu} p^2 \,, \tag{2.1}$$

the momentum space two-point function of a spin-1 current in a three-dimensional CFT has the general form

$$\langle J^\mu(p) J^\nu(-p)\rangle \equiv J^{\mu\nu}(p) = C_J \frac{\Pi^{\mu\nu}(p)}{\sqrt{p^2}} + W_J \epsilon^{\mu\nu\rho} p_\rho \,. \tag{2.2}$$

In coordinate space this takes the form

$$\langle J^\mu(x) J^\nu(0)\rangle = C_J \Pi^{\mu\nu}(i\partial)\left(\frac{1}{2\pi^2 x^2}\right) + W_J \epsilon^{\mu\nu\rho} i\partial_\rho \delta^{(3)}(x) \,. \tag{2.3}$$

The second term in (2.3) is a conformally invariant contact term that is special to three dimensions. The effect of the S-operation [13] is to transform the two parameters C_J and W_J as

$$C_J \to \frac{C_J}{C_J^2 + W_J^2} \,, \qquad W_J \to -\frac{W_J}{C_J^2 + W_J^2} \,. \tag{2.4}$$

The T-operation is a rather trivial shift in the value of W_J by an integer. The important result of [13] is that these S- and T-operations generate the modular group $SL(2,\mathbb{Z})$. In what follows we concentrate on the non-trivial S-operation.

A simple way to obtain the result (2.4) is to couple J^μ to an external vector field A_μ and integrate elementary fields (assuming the existence of an appropriate lagrangian description), to obtain the effective action

$$\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, A_\mu(p) J^{\mu\nu}(p) A_\nu(-p) + \cdots \,. \tag{2.5}$$

This would yield an effective propagator for A_μ; however $J^{\mu\nu}(p)$ is not directly invertible so if we want to proceed we should gauge fix, adding

$$\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, A_\mu(p)\left[C_J(\xi-1)\frac{p^\mu p^\nu}{\sqrt{p^2}}\right]A_\nu(-p)\,. \tag{2.6}$$

This leads to the addition of the quantity in square brackets to $J^{\mu\nu}(p)$ as

$$J_\xi^{\mu\nu}(p) = C_J\frac{1}{\sqrt{p^2}}\left(\xi p^\mu p^\nu - p^2\eta^{\mu\nu}\right) + W_J\epsilon^{\mu\nu\rho}p_\rho\,. \tag{2.7}$$

The tensor above has the inverse

$$M^\xi_{\mu\nu}(p) = \frac{1}{p^2}\left[\frac{C_J}{C_J^2+W_J^2}\frac{\Pi_{\mu\nu}(p)}{\sqrt{p^2}} - \frac{W_J}{C_J^2+W_J^2}\epsilon_{\mu\nu\rho}p^\rho + \frac{1}{C_J(\xi-1)}\frac{p_\mu p_\nu}{\sqrt{p^2}}\right], \tag{2.8}$$

which is the two-point function of A_μ. Defining then the dual conserved current as $\hat{J}^\mu(p) = i\epsilon^{\mu\nu\rho}p_\nu A_\rho(p)$ we find its two-point function to be

$$\langle \hat{J}^\mu(p)\hat{J}^\nu(-p)\rangle = \frac{C_J}{C_J^2+W_J^2}\frac{\Pi^{\mu\nu}(p)}{\sqrt{p^2}} - \frac{W_J}{C_J^2+W_J^2}\epsilon^{\mu\nu\rho}p_\rho\,. \tag{2.9}$$

This does not depend on the gauge fixing parameter ξ and it is obtained from the initial two-point function (2.2) by the S-operation (2.4).

2.2 The S-operation on the two-point function of the energy momentum tensor

In three dimensions there are two possible terms in the two-point function of a symmetric traceless and conserved rank-2 tensor

$$\langle T_{\mu\nu}(p)T_{\lambda\rho}(-p)\rangle = C_T\frac{\Pi^{(2)}_{\mu\nu,\lambda\rho}(p)}{\sqrt{p^2}} + W_T\Pi^{(1.5)}_{\mu\nu,\lambda\rho}(p)\,, \tag{2.10}$$

or, in coordinate space

$$\langle T_{\mu\nu}(x)T_{\lambda\rho}(0)\rangle = C_T\Pi^{(2)}_{\mu\nu,\lambda\rho}(i\partial)\left(\frac{1}{2\pi^2x^2}\right) + W_T\Pi^{(1.5)}_{\mu\nu,\lambda\rho}(i\partial)\,\delta^{(3)}(x)\,. \tag{2.11}$$

We have defined

$$\Pi^{(2)}_{\mu\nu,\lambda\rho}(p) = \frac{1}{2}\left[\Pi_{\mu\lambda}(p)\Pi_{\nu\rho}(p) + \Pi_{\mu\rho}(p)\Pi_{\nu\lambda}(p) - \Pi_{\mu\nu}(p)\Pi_{\lambda\rho}(p)\right], \tag{2.12}$$

$$\Pi^{(1.5)}_{\mu\nu,\lambda\rho}(p) = \frac{1}{4}\left[\epsilon_{\mu\lambda\sigma}\Pi_{\nu\rho}(p) + \epsilon_{\nu\lambda\sigma}\Pi_{\mu\rho}(p) + \epsilon_{\mu\rho\sigma}\Pi_{\nu\lambda}(p) + \epsilon_{\nu\rho\sigma}\Pi_{\mu\lambda}(p)\right]p^\sigma\,. \tag{2.13}$$

Note that the second term in (2.11) is a conformally invariant contact term special to three dimensions. If we couple $T_{\mu\nu}$ to an external field $h_{\mu\nu}$ and integrate out the elementary fields (assuming again an appropriate lagrangian formulation), we would like to invert the induced $h_{\mu\nu}$ propagator. In order to do so, we must gauge fix. A sufficiently general gauge fixing involves two arbitrary parameters ξ_1 and ξ_2 and is of the form (for clarity, we suppress in the following indices that are not necessary)

$$\langle TT\rangle \to \langle TT\rangle_\xi = M = C_T \frac{\Pi^{(2)}}{\sqrt{p^2}} + W_T \Pi^{(1.5)} + p^3 \left(\xi_1 \mathcal{A} + \xi_2 \mathcal{B}\right), \tag{2.14}$$

where it is convenient to define

$$\begin{aligned} \mathcal{A}_{\mu\nu,\lambda\rho} &= -\frac{3}{2p^4}\left(p_\mu p_\nu - \frac{1}{3}p^2\eta_{\mu\nu}\right)\left(p_\lambda p_\rho - \frac{1}{3}p^2\eta_{\lambda\rho}\right), \\ \mathcal{B}_{\mu\nu,\lambda\rho} &= \frac{1}{2p^2}\left(p_\mu p_\lambda \eta_{\nu\rho} + p_\mu p_\rho \eta_{\lambda\nu} + p_\nu p_\lambda \eta_{\mu\rho} + p_\nu p_\rho \eta_{\lambda\mu}\right) - \frac{4}{p^4} p_\mu p_\nu p_\lambda p_\rho . \end{aligned} \tag{2.15}$$

The inverse of (2.14) may now be computed and we find

$$p^6 M^{-1} = \frac{C_T}{C_T^2 + W_T^2}\frac{\Pi^{(2)}}{\sqrt{p^2}} - \frac{W_T}{C_T^2 + W_T^2}\Pi^{(1.5)} + p^3\left(\frac{1}{\xi_1}\mathcal{A} + \frac{1}{\xi_2}\mathcal{B}\right). \tag{2.16}$$

This is the propagator of the field $h_{\mu\nu}$ which has zero scaling dimension. If we define now a symmetric traceless and conserved tensor in momentum space as

$$\hat{T}_{\mu\nu} = \Pi^{(1.5)}_{\mu\nu,\lambda\rho} h^{\lambda\rho}, \tag{2.17}$$

we find that

$$\langle \hat{T}_{\mu\nu}\hat{T}_{\lambda\rho}\rangle = \frac{C_T}{C_T^2 + W_T^2}\frac{\Pi^{(2)}_{\mu\nu,\lambda\rho}(p)}{\sqrt{p^2}} - \frac{W_T}{C_T^2 + W_T^2}\Pi^{(1.5)}_{\mu\nu,\lambda\rho}(p). \tag{2.18}$$

We see that (2.18) does not depend on the gauge fixing parameters and is of the form (2.10), but with the coefficients transformed by $S \in \mathrm{SL}(2,\mathbb{Z})$.

2.3 Generalization to higher-spin currents

The computations presented in the previous sections for $s = 1, 2$ can be generalized to higher-spin conserved currents. In three dimensions, there are two possible conformally invariant tensor structures that may appear in the momentum space two-point function of a spin-s conserved current (refer to the latter as $T_{(s)}$ and suppress indices)

$$\langle T_{(s)} T_{(s)}\rangle = C_s \frac{\Pi^{(s)}(p)}{\sqrt{p^2}} + W_s \Pi^{(s-1/2)}(p). \tag{2.19}$$

The first term in (2.19) is the usual term (see for example [24]) that corresponds to an appropriately symmetrized and traceless product of (2.1), while the second term is a conformally invariant contact term. These terms have the schematic form

$$\Pi^{(s)} \sim \left(\Pi^{(1)}\right)^s, \qquad \Pi^{(s-1/2)} \sim \left(\Pi^{(1)}\right)^{s-1}\Pi^{(0.5)}, \tag{2.20}$$

where by $\Pi^{(0.5)}, \Pi^{(1)}$ we mean $\Pi^{(0.5)}_{\mu\nu} = \epsilon_{\mu\nu\lambda}p^\lambda$ and $\Pi^{(1)}_{\mu\nu} = \Pi_{\mu\nu}$. The index structure of each tensor in (2.20) is *uniquely* determined by symmetry and tracelessness, while the requirement of canonical dimensions $\Delta = s + 1$ also implies conservation. By coupling to an external field $h_{(3-s)}$ and gauge fixing in order to invert its propagator, one finally arrives at the gauge independent two-point function

$$\langle \hat{T}_{(s)} \hat{T}_{(s)} \rangle = \frac{C_s}{C_s^2 + W_s^2} \frac{\Pi^{(s)}(p)}{\sqrt{p^2}} - \frac{W_s}{C_s^2 + W_s^2} \Pi^{(s-1/2)}(p) \,, \tag{2.21}$$

for the dual current

$$\hat{T}_{(s)} = \Pi^{(s-1/2)} h_{(3-s)} \,. \tag{2.22}$$

3. Double-trace deformations and the S-operation

As a first step towards understanding our results so far, we ask if the $\mathrm{SL}(2,\mathbb{Z})$ action, and in particular the S-operation, have an underlying dynamics and whether this dynamics is generic in three-dimensional CFTs. Rather surprisingly, we find that dynamically induced duality transformations are common in three-dimensional CFTs. Moreover, the underlying dynamics appears to be rather generic and amounts to double-trace deformations. Indeed, we are able to explicitly demonstrate that the S-operation on spin-1 and spin-2 conserved currents can be induced by certain irrelevant double-trace deformations, under the assumption that these lead to well-defined UV fixed-points. Therefore, we see the formation of an interesting pattern on the space of three-dimensional CFTs.

3.1 A precursor to the S-operation

To set the stage, we recall here some of the salient properties of two well-known three-dimensional theories with non-trivial (large-N) fixed points: the critical $O(N)$ Vector model and Gross-Neveu models. The corresponding (euclidean) actions are

$$\mathcal{L}_{O(N)} = \int d^3x \left[\frac{1}{2} (\partial\phi)^2 + \frac{1}{2}\sigma \left(\phi^2 - \frac{1}{g} \right) \right] , \tag{3.1}$$

$$\mathcal{L}_{GN} = - \int d^3x \left[\bar{\psi} \partial\!\!\!/ \psi + \frac{G}{2} (\bar{\psi}\psi)^2 \right] , \tag{3.2}$$

for the N-component scalar $\phi(x)$ and Majorana fermion $\psi(x)$. The auxiliary scalar $\sigma(x)$ enforces the constraint in the $O(N)$ Vector model. Both models are renormalizable in the $1/N$ expansion and have non-trivial large-N fixed points corresponding to strongly coupled CFTs where the elementary fields $\phi(x)$ and $\psi(x)$ acquire anomalous dimensions of order $1/N$. Most importantly, and this is the key to the $1/N$ renormalizability of the models, the composite operators $\phi^2(x)$ and $(\bar{\psi}\psi)(x)$ receive large corrections to their scaling dimensions.

We may interpret this latter phenomenon as a duality transformation in the following sense. In a three-dimensional CFT all quasi-primary operators must transform under unitary irreducible representations (UIR) of $\mathrm{SO}(3,2)$. For example, the bilinear leading-twist[4] operators in the $O(N)$-singlet sector of the free field theory limit of the $O(N)$ Vector model, realize the UIRs denoted by $\chi = [\Delta_0, s]$ where Δ_0 is the *canonical* scaling dimension and s the total spin. In particular, we note that the composite operator $\phi^2(x)$ realizes the

[4] Operators containing more than two $\phi^a(x)$'s are *higher-twist*.

JHEP12(2003)020

UIR $[1,0]$ [23]. A similar classification must also hold true for the quasi-primary operators of the corresponding sector at the large-N fixed point of the model. Explicit calculations then show that almost all leading-twist quasi-primary operators of the non-trivial CFT realize UIRs of the form $[\Delta_0 + o(1/N), s]$ [25, 26]. The important exception is connected with the large-N realization of the operator $\phi^2(x)$. In the non-trivial CFT, the UIR $[1,0]$ does not exist and instead one finds the UIR $[2+o(1/N),0]$. Therefore, in the strict large-N limit this particular sector of the operator spectrum in the non-trivial CFT is obtained from the spectrum of the free CFT by changing the UIR $[1,0]$ to the UIR $[2,0]$. As is well known, these two UIRs are equivalent and are exchanged by a Weyl reflection [27, 28] in $SO(3,2)$.[5] In that sense, the non-trivial large-N fixed point of the $O(N)$ Vector model may be viewed as the result of a 'duality' transformation on the spectrum of the free CFT. In exactly the same way, a particular sector in the spectrum of the non-trivial fixed point of the $O(N)$ Gross-Neveu model is obtained, in the strict large-N limit, by implementing a duality operation on the spectrum of the free fermionic CFT. In that case one changes the UIR $[2,0]$ that is realized by the composite operator $(\bar{\psi}\psi)(x)$ to its shadow UIR $[1,0]$.

The above duality operation may be viewed as a precursor to the S-operation acting on two-point functions of scalars. Moreover, we have a good understanding of the dynamics underlying this operation which is similar in both models. Indeed, we know that the non-trivial fixed-point in each model is reached under the influence of a certain double-trace deformation. In the case of the $O(N)$ Vector model, the deformation is relevant and the corresponding fixed-point is in the IR, while in the case of the Gross-Neveu model the deformation is irrelevant and reveals the existence of a non-trivial UV fixed point. Let us see how such generic double-trace deformations induce the duality operation that we have mentioned. Consider a three-dimensional euclidean CFT with elementary fields ϕ and partition function

$$\mathcal{Z} = \int (\mathcal{D}\phi) e^{-S(\phi)} \,. \tag{3.3}$$

Consider now a scalar composite operator $\mathcal{O}(x)$ with canonical dimension $1/2 < \Delta < 3/2$. We deform the action by

$$-\frac{f}{2}\int d^3x\, \mathcal{O}^2(x)\,, \tag{3.4}$$

and ask for the two-point function of $\mathcal{O}(x)$ at the IR fixed point where the deformation (3.4) presumably leads the theory (3.3). We proceed by a direct calculation

$$\begin{aligned}\left\langle \mathcal{O}(x_1)\mathcal{O}(x_2) e^{\frac{f}{2}\int d^3x\, \mathcal{O}^2(x)}\right\rangle &= \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_f \\ &= \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_0 + \frac{f}{2}\int d^3x \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}^2(x)\rangle_0 + \\ &\quad + \frac{f^2}{8}\int d^3x d^3y \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}^2(x)\mathcal{O}^2(y)\rangle_0 + \cdots \,. \end{aligned} \tag{3.5}$$

[5] In general, a Weyl reflection exchanges the UIRs $\chi = [\Delta, s]$ and $\tilde{\chi} = [d-\Delta, s]$. These are called *shadow* UIRs and have the same Casimirs. The conformally invariant two-point function corresponding to the UIR χ is then defined as an intertwiner between the UIRs χ and $\tilde{\chi}$. The symmetry between the two is broken in a given CFT by the Ward identities which pick one UIR to be "elementary".

JHEP12(2003)020

We can use the OPE of $\mathcal{O}(x)$ with itself to calculate the correlation functions in (3.5). Moreover, we assume now the existence of a suitable large-N expansion such that the leading N contribution comes from the two-point function $\langle\mathcal{O}\mathcal{O}\rangle$. Then, taking into account the combinatorics we find

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_f = \langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_0 + f\int d^3x \langle\mathcal{O}(x_1)\mathcal{O}(x)\rangle_0\langle\mathcal{O}(x)\mathcal{O}(x_2)\rangle_f + \cdots\,, \tag{3.6}$$

where the dots contain terms subleading in $1/N$ that we drop. Denoting the momentum space two-point function by $Q(p)$, we obtain from (3.6)

$$Q_f(p) = \frac{Q_0(p)}{1 - fQ_0(p)}\,. \tag{3.7}$$

The assumption of the existence of a non-trivial fixed-point enters now in a crucial way. Indeed, the coupling has dimension $[mass]^{3-2\Delta}$ and sets the scale of physical processes. To study the IR behavior of (3.7) we need to assume that f can be made finite by renormalization and expand for small momenta p as

$$f^2Q_f(p) = -\frac{f}{\left(1 - \frac{1}{fQ_0(p)}\right)} = -f - Q_0^{-1}(p) + \cdots\,, \qquad Q_0(p) \approx \frac{1}{p} \gg 1\,. \tag{3.8}$$

This is the properly normalized two-point function of the operator $\mathcal{O}(x)$ for mass scales less than the scale set by the renormalized coupling. The minus sign in the second term on the rhs of (3.8) guarantees the positivity of the x-space two-point function in the IR, while the first term gives a conformally non-invariant contact term in x-space and should be dropped. We see then that the effect of the deformation (3.4) is the duality operation $Q_0(p) \to f^2\, Q_f(p) \approx -Q_0^{-1}(p)$, or in an algebraic sense to change the UIR $[1,0]$ to $[2,0]$. This is very similar to the "first-half" of the S-operation (e.g. going from J^μ to A_μ for $W = 0$), discussed in the previous section.

In a similar fashion we may consider theories having a scalar operator $\Psi(x)$ with dimension $3/2 < \Delta < 3$ and deform by the irrelevant double-trace deformation

$$-\frac{G}{2}\int d^3x \Psi(x)\Psi(x)\,. \tag{3.9}$$

In this case one expects that the initial theory may be the IR limit of a non-trivial UV fixed point, as in the explicit example of the Gross-Neveu model. As before, we assume a large-N expansion for the correlation functions involving $\Psi(x)$ and $\Psi^2(x)$ and by a direct calculation we obtain

$$\mathcal{J}_G(p) = \frac{\mathcal{J}_0(p)}{1 - G\mathcal{J}_0(p)}\,, \tag{3.10}$$

where $\mathcal{J}(p)$ denotes the momentum space two-point function of $\Psi(x)$. The coupling G has negative mass dimension and as before we have to assume that it can be made finite by renormalization. Then we can study the large momentum behavior of the properly normalized two-point function of $\Psi(x)$ for momenta much larger than the scale set by the renormalized coupling as

$$G^2\mathcal{J}_G(p) = -\frac{G}{\left(1 - \frac{1}{G\mathcal{J}_0(p)}\right)} = -G - \mathcal{J}_0^{-1}(p) + \cdots\,, \qquad -\mathcal{J}_0(p) \approx \sqrt{p^2} \gg 1\,. \tag{3.11}$$

As before, up to a contact term this gives a positive definite x-space UV two-point function. Therefore the irrelevant deformation (3.9) produces the duality operation $\mathcal{J}_0(p) \to G^2\mathcal{J}_G(p) \approx -\mathcal{J}_0^{-1}(p)$ and is also very similar to the "first-half" of the S-operation of the previous section.

An important property of the UIRs $[1,0]$ and $[2,0]$ is the fact that they are both above the unitarity bound[6] for UIRs of SO(3,2) [30]. This is exceptional, as the same is no longer true if one wished to perform the same duality operation on most of the UIRs of the free CFT. In particular, all the shadows of the UIRs $[s+1,s]$ that correspond to the infinite set of conserved currents in the free CFT fall below the unitarity bound. Nevertheless, there exist explicit examples of three-dimensional theories where the puzzle is apparently resolved. These examples correspond to fermionic theories with action of the generic form

$$S = -\int d^3x \left[\bar{\psi}\partial\!\!\!/\psi + \frac{F}{2}(\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma_\mu\psi)\right], \tag{3.12}$$

for the N-component Majorana spinor $\psi(x)$. The conserved U(1) current is $J_\mu(x) = \bar{\psi}(x)\gamma_\mu\psi(x)$ and while the interaction in (3.12) appears non-renormalizable (irrelevant) by power counting, such models are renormalizable in the $1/N$ expansion and also have non-trivial UV fixed points that correspond to gauge CFTs [29, 31, 32, 33, 13]. In contrast to the Gross-Neveu model the bare coupling constant F is not renormalized.

The puzzle with the duality operation is resolved in an interesting way in these models [34]. The shadow operator of the U(1) current must be regarded as a gauge field. Indeed, this is the main finding of studies in these models and has led to the understanding that the UV fixed-point is related to conformal QED_3. Gauge fields are not required to realize UIRs of the conformal group; only gauge invariant quantities do. On the other hand, given a three-dimensional gauge field $A_\mu(x)$, one can construct the gauge invariant conserved current

$$\hat{J}^\mu(x) = \epsilon^{\mu\nu\rho}\partial_\nu A_\rho(x)\,. \tag{3.13}$$

which realizes the standard UIR $[2,1]$ of SO(3,2). The lengthy discussion above begins to clarify. The duality operation on the spin-1 current of the free theory corresponds to the "first-half" of the S-operation and will lead to the gauge field. The construction (3.13) provides the "second-half" of the S-operation and one finally gets the spin-1 current of the non-trivial theory. The underlying dynamics is a double-trace deformation. We demonstrate this in the next subsection.

3.2 Double-trace deformations and the S-operation on the two-point function of the spin-1 current

Consider a three-dimensional CFT with a U(1) current $J_\mu(x)$ having momentum space two-point function (2.2). We perturb the theory by the irrelevant deformation

$$-\frac{f}{2}\int \mathrm{d}^3x J_\mu(x)J_\mu(x)\,. \tag{3.14}$$

[6]The unitarity bound for the scaling dimensions in SO(3,2) is $\Delta \geq 1/2$ for spinless irreps and $\Delta \geq s+1$ for irreps with spin $s \geq 1$.

To proceed with a calculation similar to (3.5) we assume a $1/N$ expansion of the correlation functions involving $J_\mu(x)$ and $J_\mu^2(x)$, with leading terms coming from the two-point function of $J_\mu(x)$. We then obtain

$$\left[\delta^\mu{}_\rho - f J_0^\mu{}_\rho(p)\right] J_f^{\rho\nu}(p) = J_0^{\mu\nu}(p) \,. \tag{3.15}$$

As before, the existence of a non-trivial UV fixed point implies that the solution for large momenta is of the form

$$f^2 \, J_f^{\rho\nu}(p) \simeq -f\,\delta^{\rho\nu} - (J_0^{-1})^{\rho\nu} + \cdots \,, \tag{3.16}$$

which is similar to (3.8) and (3.11). In the present case, however, we need to gauge fix so that J_0^{-1} exists. Alternately, we may simply write the ansatz

$$f^2 \, J_f^{\mu\nu}(p) = \hat{C}_J \frac{\Pi^{\mu\nu}(p)}{\sqrt{p^2}} + \hat{W}_J \epsilon^{\mu\nu\rho} p_\rho \tag{3.17}$$

and using (3.15) we easily find[7]

$$\hat{C}_J = \frac{f^2[C_J + fp(C_J^2 + W_J^2)]}{1 + 2fpC_J + f^2p^2(C_J^2 + W_J^2)} \,, \tag{3.18}$$

$$\hat{W}_J = \frac{f^2 \, W_J}{1 + 2fpC_J + f^2p^2(C_J^2 + W_J^2)} \,. \tag{3.19}$$

We want to study this for large momenta (in the UV), assuming at the same time a finite critical coupling f. One obtains,

$$\hat{C}_J \sim \frac{f}{\sqrt{p^2}} - \frac{1}{p^2}\frac{C_J}{C_J^2 + W_J^2} + \cdots \tag{3.20}$$

$$\hat{W}_J \sim \frac{1}{p^2}\frac{W_J}{C_J^2 + W_J^2} + \cdots \,. \tag{3.21}$$

The first term on the rhs of (3.20) gives a non-conformally-invariant contact term which is dropped. The two-point function (3.17) can be viewed as the properly normalized two-point function of a vector operator with dimension 1 and to avoid problems with the unitarity bound we must assume that this is a gauge field. Then the two-point function of the associated gauge invariant conserved current, which we write in momentum space as $\hat{J}^\mu(p) = i\epsilon^{\mu\nu\rho} p_\nu A_\rho(p)$, is found to be

$$\hat{J}^{\mu\nu}(p) = \frac{C_J}{C_J^2 + W_J^2}\frac{\Pi^{\mu\nu}(p)}{\sqrt{p^2}} - \frac{W_J}{C_J^2 + W_J^2}\epsilon^{\mu\nu\rho} p_\rho \,. \tag{3.22}$$

We see that (3.22) is obtained from (2.2) by the S-operation (2.4).

[7]Similar formulas were derived also in [22].

3.3 Double-trace deformations and the S-operation on the two-point function of the energy momentum tensor

That our previous calculations work out is perhaps not surprising, since our lengthy discussion of the various well-known three-dimensional CFTs unveils a similar underlying dynamics. What is rather surprising is that a similar double-trace deformation appears also to be the underlying dynamics of the S-operation on the two-point function of the energy momentum tensor. Consider the irrelevant deformation

$$-\frac{g}{2}\int d^3x\ T_{\mu\nu}(x)T_{\mu\nu}(x)\,. \tag{3.23}$$

Under the same assumptions as before, namely the existence of a suitable large-N limit for correlation functions and a non-trivial UV fixed point, following similar calculations we obtain

$$\left[\delta_{\mu\alpha}\delta_{\nu\beta} - gT^0_{\mu\nu,\alpha\beta}(p)\right] T^g_{\alpha\beta,\rho\sigma}(p) = T^0_{\mu\nu,\rho\sigma}(p)\,. \tag{3.24}$$

This gives for large momenta

$$g^2\, T^g_{\mu\nu,\rho\sigma}(p) \approx g^2\, \delta_{\mu\nu}\delta_{\rho\sigma} - (T_0^{-1})_{\mu\nu,\rho\sigma}(p) + \cdots\,. \tag{3.25}$$

Again we need to gauge fix such that T_0^{-1} exists, but with our previous experience we try an ansatz of the form

$$g^2\, T^g_{\mu\nu,\rho\sigma}(p) = \hat{C}_T\frac{\Pi^{(2)}_{\mu\nu,\lambda\rho}(p)}{\sqrt{p^2}} + \hat{W}_T\Pi^{(1.5)}_{\mu\nu,\lambda\rho}(p)\,, \tag{3.26}$$

After some lengthy but straightforward algebra we obtain

$$\hat{C}_T = \frac{g^2[C_T - gp^3(C_T^2+W_T^2)]}{1+(gp^3)^2(C_T^2+W_T^2) - 2gp^3C_T}\,, \tag{3.27}$$

$$\hat{W}_T = \frac{g^2\, W_T}{1+(gp^3)^2(C_T^2+W_T^2) - 2gp^3C_T}\,, \tag{3.28}$$

in complete analogy with (3.20) and (3.21). The result is a two-point function of a dimension zero symmetric and traceless tensor $h_{\mu\nu}$ which, as before, we must require to be a gauge field. Gauge symmetry in this case may be local diffeomorphisms and $h_{\mu\nu}$ may be viewed as the symmetric traceless part of the metric tensor in three-dimensional gravity. Then, the gauge invariant symmetric, traceless and conserved tensor of dimension three

$$\hat{T}_{\mu\nu}(p) = i\Pi^{(1.5)}_{\mu\nu,kl}h_{kl}(p)\,, \tag{3.29}$$

has two-point function exactly of the form (2.18). Clearly, it would be of interest to find an explicit theory where this is implemented.

One can also straightforwardly show that an irrelevant double-trace deformation of a similar form induces the S-operation on the two-point functions of higher-spin conserved currents. We leave this for future work.

4. The bulk view

Witten has suggested [13] that the action of $\mathrm{SL}(2,\mathbb{Z})$ on the correlators of abelian currents can be understood as the holographic image of electromagnetic duality of a U(1) gauge theory on AdS_4. Specifically, he suggested that the S-operation corresponds to a choice of boundary condition. Given our results for the energy momentum tensor two-point function, we can ask if at least in some approximation, there is a natural place for $\mathrm{SL}(2,\mathbb{Z})$ in a four-dimensional gravity theory. Let us first consider the case of gravity in the AdS_4 bulk. It is convenient to write the Einstein-Hilbert action in MacDowell-Mansouri form [18]. To do so, we formally introduce an $SO(3,2)$ connection $\omega^A{}_B = -\omega^B{}_A$ with curvature $\mathcal{R}^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B$. An appropriate action is of the form [35]

$$I_{MM} = -\frac{1}{4L}\int_{\mathcal{M}} V^A \mathcal{R}^{BC}\mathcal{R}^{DE}\epsilon_{ABCDE}\,, \quad \epsilon_{-10123}=1\,, \qquad \eta_{AB} = (-+++-)\,, \tag{4.1}$$

where V^A is a $SO(3,2)$ vector satisfying $V^A V_A = -L^2$. This is actually the defining condition for AdS_4 with radius L embedded into a five-dimensional space with metric η_{AB}. The gauge choice $V^{-1} = L, V^a = 0$, $a = 0,1,2,3$ splits the connection into the $SO(3,1)$ connection and the vierbein as

$$\omega^A{}_B \quad \to \quad \omega^a{}_b\,, \quad \omega^a{}_{-1} = \frac{1}{L}e^a\,. \tag{4.2}$$

The curvature then splits into two pieces

$$\mathcal{R}^a{}_b = R^a{}_b + \frac{1}{L^2}e^a \wedge e_b\,, \tag{4.3}$$

$$\mathcal{R}^a{}_{-1} = \frac{1}{L}de^a + \frac{1}{L}\omega^a{}_b \wedge e^b = T^a\,, \tag{4.4}$$

the latter being the torsion. The MacDowell-Mansouri action then reduces to

$$\begin{aligned} I_{MM} &= -\frac{1}{2L^2}\int_{\mathcal{M}} e^a\wedge e^b \wedge R^{cd}\epsilon_{abcd} - \frac{1}{4L^4}\int_{\mathcal{M}} e^a\wedge e^b\wedge e^c\wedge e^d\epsilon_{abcd} + \frac{1}{2}\int_{\mathcal{M}} \mathrm{tr}\, R\wedge\tilde{R} \\ &= \frac{1}{L^2}\int_{\mathcal{M}} d^4x\sqrt{-g}\,\left(R + \frac{6}{L^2}\right) - 8\pi^2\chi(\mathcal{M})\,, \end{aligned} \tag{4.5}$$

where $\tilde{R}^{ab} = \frac{1}{2}\epsilon^{abcd}R_{cd}$ and $\chi(\mathcal{M})$ is the Euler character. (Of course we are being notationally loose here: AdS is not compact, but we will discuss boundary terms presently). In fact, the Euler character is of little interest for the present discussion: it is topological, and the quadratic boundary terms that it creates are divergent and are removed by boundary counterterms. On the other hand, it is well-known that the Einstein-Hilbert term leads to the standard C_T term in the two-point function of the energy momentum tensor in the boundary theory [36].

There is however another topological term that is of interest to our discussion. It is the (bulk part of the) Pontryagin class

$$P(M) = -\frac{1}{8\pi^2}\int_{\mathcal{M}} \mathcal{R}^A{}_B\mathcal{R}^B{}_A = -\frac{1}{8\pi^2}\int_{\mathcal{M}} \mathrm{Tr}\, R\wedge R\,, \tag{4.6}$$

the latter equality holding up to torsion terms on the boundary. As we stated, this term is topological and gives a boundary contribution

$$P(\mathcal{M}) = -\frac{1}{8\pi^2}\int_{\partial\mathcal{M}} \mathrm{Tr}\left(\omega\wedge d\omega + \frac{2}{3}\omega\wedge\omega\wedge\omega\right). \tag{4.7}$$

To evaluate (4.7) we expand the metric to linearized level around the AdS background

$$ds^2 = \bar{g}_{ab}dy^a dy^b = \frac{L^2}{r^2}\left(dr^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right), \quad \mu,\nu = 1,2,3 \tag{4.8}$$

as $g_{ab} = \bar{g}_{ab} + r^{-2}\,h_{ab}$. Substituting this back to (4.7) we obtain

$$P(\mathcal{M}) = -\frac{1}{16\pi^2}\int d^3x\ \epsilon^{\mu\nu\lambda} h_\mu{}^\epsilon{}_{,\sigma}\left(h_\lambda{}^\sigma{}_{,\epsilon} - h_{\lambda\epsilon}{}^{,\sigma}\right)_{,\nu}, \tag{4.9}$$

where $h^{\mu\nu}(\bar{x})$ is the boundary value of the metric fluctuation and indices have been raised and lowered with $\eta_{\mu\nu}$. In momentum space, we find

$$P(\mathcal{M}) = \frac{i}{16\pi^2}\int \frac{d^3k}{(2\pi)^3} h^{\mu\epsilon}(k)\Pi^{(1.5)}_{\mu\epsilon,\lambda\sigma}(k)h^{\lambda\sigma}(-k). \tag{4.10}$$

Clearly, this gives the contribution to the energy momentum tensor two-point function proportional to W_T.

The above results lead us to consider the following euclidean bulk action

$$S_{bulk} = \frac{1}{4\pi}\left[\frac{4\pi}{g_N^2}\int_{\mathcal{M}} \mathrm{Tr}\,\mathcal{R}\wedge\tilde{\mathcal{R}} + i\frac{\theta_N}{2\pi}\int_{\mathcal{M}} \mathrm{Tr}\,\mathcal{R}\wedge\mathcal{R}\right] + \int_{\partial\mathcal{M}} \mathcal{L}_{\text{c.t.}}, \tag{4.11}$$

where we have introduced a dimensionless coupling g_N which in the standard normalization of the Einstein-Hilbert action is

$$\frac{4\pi}{g_N^2} = \frac{L^2}{8G_4}, \tag{4.12}$$

and a theta-angle θ_N. The action (4.11) expanded around the AdS_4 background gives rise to the boundary two-point function (2.10). The parameters C_T and W_T depend respectively on g_N and θ_N.

Now we see that the shift of $\theta_N \to \theta_N + 2\pi$ induces a shift of W_T in the boundary theory: this is the T-operation. On the other hand, the action has been deliberately written in a form which is reminiscent of gauge theory. In particular, we can also write the action in the form

$$\frac{1}{4\pi}\sum_{\pm}\tau_\pm\,\mathrm{Tr}\,\mathcal{R}^\pm\wedge\mathcal{R}^\pm, \tag{4.13}$$

if we introduce

$$\mathcal{R}^\pm = \frac{1}{2}\left(\mathcal{R}\pm\tilde{\mathcal{R}}\right), \tag{4.14}$$

$$\tau_\pm = \frac{4\pi}{g_N^2} \pm i\,\frac{\theta_N}{2\pi}. \tag{4.15}$$

JHEP12(2003)020

We see now that a least at the linearized level (i.e. if we consider only two-point functions in the boundary) one expects that the S-transformation in the boundary is induced by an appropriate generalization of electric-magnetic duality transformation for the bulk gravity.

The generalization of our results for higher spins now looks straightforward at the linearized level. One may start from the action written down by Vasiliev in [21] and add to it the appropriate generalization of the Pontryagin term. It appears then possible that one can arrive at an action that can be written in the form of (4.13), but we leave this discussion for the future.

5. Discussion

There is a number of potentially interesting implications of our results. On a broader level, they indicate an intimate relation between $\mathrm{SL}(2,\mathbb{Z})$ duality and the holography of free field theories. In this context, it would be of interest to find examples where the sort of duality that we have discussed here for spin ≥ 2 is implemented. Certain Higher-Spin theories that have been constructed on AdS_4 are a natural place for such studies.

One can also speculate as to whether or not there are theories for which the duality operates beyond the linearized level. For this one should study three-point functions of spin-1, spin-2 and perhaps higher-spin currents in order to understand whether there exist some natural discrete action in three-dimensional CFTs beyond the linearized level. Since conformal invariance fixes the form of three-point functions up to a number of constant parameters, it is possible to follow the calculations in this work and evaluate the three-point function of the dual (non-abelian) spin-1, spin-2 and higher-spin currents.

An intriguing and potentially far reaching implication of our results is the possibility that many known three-dimensional CFTs can provide information for quantum field theory on $(\mathrm{A})\mathrm{dS}_4$, even without the involvement of supersymmetry. At this point we note that the parameter N of the boundary CFTs is related to the Planck length of the bulk theory. In this sense it is perhaps not inconceivable that the wealth of statistical physics phenomena could be directly related to as yet undiscovered phenomena of quantum fields (or strings, membranes), on $(\mathrm{A})\mathrm{dS}_4$. As an example, one might wonder whether black hole-like configurations in the bulk are related to thermal effects in the boundary.

Acknowledgments

We thank L. Andrianopoli, L. Alvarez-Gaumé, E. Floratos, A. Kovner, C. A. Lütken, D. Minic, Y. Oz, M. Porrati, P. Vanhove and M. Vasiliev for useful discussions. The work of RGL is supported in part by the U.S. Department of Energy under contract DE-FG02-91ER40677. RGL would also like to thank the Aspen Center for Physics for a stimulating environment where some of this work was carried out.

References

[1] M. Moshe and J. Zinn-Justin, *Quantum field theory in the large-N limit: a review*, *Phys. Rept.* **385** (2003) 69 [hep-th/0306133].

[2] R. Gopakumar, *From free fields to AdS*, hep-th/0308184.

[3] M. Bianchi, J.F. Morales and H. Samtleben, *On stringy $AdS_5 \times S^5$ and higher spin holography*, *J. High Energy Phys.* **07** (2003) 062 [hep-th/0305052].

[4] B. Sundborg, *Stringy gravity, interacting tensionless strings and massless higher spins*, *Nucl. Phys.* **102** *(Proc. Suppl.)* (2001) 113 [hep-th/0103247].

[5] I.R. Klebanov and A.M. Polyakov, *AdS dual of the critical O(N) vector model*, *Phys. Lett.* **B 550** (2002) 213 [hep-th/0210114].

[6] E. Sezgin and P. Sundell, *Massless higher spins and holography*, *Nucl. Phys.* **B 644** (2002) 303 [hep-th/0205131], erratum *ibid.* **B660** (2003) 403.

[7] M.A. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in $(A)ds_d$*, *Phys. Lett.* **B 567** (2003) 139 [hep-th/0304049].

[8] D. Francia and A. Sagnotti, *On the geometry of higher-spin gauge fields*, *Class. and Quant. Grav.* **20** (2003) S473 [hep-th/0212185].

[9] L. Girardello, M. Porrati and A. Zaffaroni, *3-d interacting cfts and generalized Higgs phenomenon in higher spin theories on AdS*, *Phys. Lett.* **B 561** (2003) 289 [hep-th/0212181].

[10] A.C. Petkou, *Evaluating the AdS dual of the critical O(N) vector model*, *J. High Energy Phys.* **03** (2003) 049 [hep-th/0302063].

[11] R.G. Leigh and A.C. Petkou, *Holography of the $N = 1$ higher-spin theory on AdS_4*, *J. High Energy Phys.* **06** (2003) 011 [hep-th/0304217].

[12] E. Sezgin and P. Sundell, *Holography in 4d (super) higher spin theories and a test via cubic scalar couplings*, hep-th/0305040.

[13] E. Witten, $\mathrm{SL}(2,\mathbb{Z})$ *action on three-dimensional conformal field theories with abelian symmetry*, hep-th/0307041.

[14] C.A. Lutken and G.G. Ross, *Duality in the quantum Hall system*, *Phys. Rev.* **B 45** (1992) 11837.

[15] C.A. Lutken, *Geometry of renormalization group flows constrained by discrete global symmetries*, *Nucl. Phys.* **B 396** (1993) 670.

[16] E.H. Fradkin and S. Kivelson, *Modular invariance, self-duality and the phase transition between quantum Hall plateaus*, *Nucl. Phys.* **B 474** (1996) 543 [cond-mat/9603156].

[17] A. Kapustin and M.J. Strassler, *On mirror symmetry in three dimensional abelian gauge theories*, *J. High Energy Phys.* **04** (1999) 021 [hep-th/9902033].

[18] S.W. MacDowell and F. Mansouri, *Unified geometric theory of gravity and supergravity*, *Phys. Rev. Lett.* **38** (1977) 739, erratum *ibid.* **38** (1977) 1376.

[19] H. Garcia-Compean, O. Obregon and C. Ramirez, *Gravitational duality in MacDowell-Mansouri gauge theory*, *Phys. Rev.* **D 58** (1998) 104012 [hep-th/9802063].

[20] C.M. Hull, *Duality in gravity and higher spin gauge fields*, *J. High Energy Phys.* **09** (2001) 027 [hep-th/0107149].

[21] M.A. Vasiliev, *Cubic interactions of bosonic higher spin gauge fields in AdS_5*, *Nucl. Phys.* **B 616** (2001) 106 [hep-th/0106200].

[22] C.P. Burgess and B.P. Dolan, *Particle-vortex duality and the modular group: applications to the quantum Hall effect and other 2-d systems*, hep-th/0010246.

[23] C. Fronsdal, *Elementary particles in a curved space, 4. Massless particles*, *Phys. Rev.* **D 12** (1975) 3819.

[24] H. Osborn and A.C. Petkou, *Implications of conformal invariance in field theories for general dimensions*, *Ann. Phys. (NY)* **231** (1994) 311 [hep-th/9307010].

[25] K. Lang and W. Ruhl, *The critical O(N) sigma model at dimensions $2 < d < 4$: a list of quasiprimary fields*, *Nucl. Phys.* **B 402** (1993) 573.

[26] A. Petkou, *Conserved currents, consistency relations and operator product expansions in the conformally invariant O(N) vector model*, *Ann. Phys. (NY)* **249** (1996) 180 [hep-th/9410093].

[27] K. Koller, *Commun. Math. Phys.* **40** (1975) 15.

[28] V.K. Dobrev, *Intertwining operator realization of the AdS/CFT correspondence*, *Nucl. Phys.* **B 553** (1999) 559 [hep-th/9812194].

[29] G. Parisi, *The theory of nonrenormalizable interactions, 1. The large N expansion*, *Nucl. Phys.* **B 100** (1975) 368.

[30] N.T. Evans, *Discrete series for the universal covering group of the $3+2$ dimensional de Sitter group*, *J. Math. Phys.* **8** (1967) 170.

[31] S. Hands, *0 (1/N(f) corrections to the thirring model in* $2 < d < 4$, *Phys. Rev.* **D 51** (1995) 5816 [hep-th/9411016].

[32] D. Anselmi, *Large-N expansion, conformal field theory and renormalization-group flows in three dimensions*, *J. High Energy Phys.* **06** (2000) 042 [hep-th/0005261].

[33] V. Borokhov, A. Kapustin and X. Wu, *Topological disorder operators in three-dimensional conformal field theory*, *J. High Energy Phys.* **11** (2002) 049 [hep-th/0206054].

[34] E.S. Fradkin and M.Y. Palchik, *Conformal quantum field theory in D-dimensions*, Kluwer Academic Publishers 1996.

[35] K.S. Stelle and P.C. West, *Spontaneously broken de Sitter symmetry and the gravitational holonomy group*, *Phys. Rev.* **D 21** (1980) 1466.

[36] H. Liu and A.A. Tseytlin, *$D = 4$ super Yang-Mills, $D = 5$ gauged supergravity and $D = 4$ conformal supergravity*, *Nucl. Phys.* **B 533** (1998) 88 [hep-th/9804083].

[37] E. Witten, *On S duality in abelian gauge theory*, *Selecta Math.* **1** (1995) 383 [hep-th/9505186].

JHEP12(2003)020